Shifted-Frequency Internal Equivalence

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***Abstract*— The shifted frequency internal equivalence (SFIE) theorem involving inhomogeneous regions is introduced and proven. For a lossless inhomogeneous region using a vector Green's theorem and potential formulation, it is shown that the frequency-domain electromagnetic field at frequency ω inside the region can be obtained using a set of equivalent volume and surface currents radiating in free space and at the different frequency ω_0. The equivalent currents thus obtained are functions of the two frequencies, electric- and magnetic-volume-type sources of the original problem, material parameters, and the original field phasors at ω, and they only exist inside the region and on its boundary. A direct application of this equivalence is that it can be used to construct an internal equivalence at a shifted frequency for electromagnetic scattering problems if data are needed in a band of frequency. ω_0 can be kept constant while the incident field frequency changes and, as a result, full computation of fields at each different frequency for volume-type equivalent sources can be avoided.**

***Index Terms*— Electromagnetic analysis, electromagnetic scattering by nonhomogeneous media, frequency-domain analysis.**

I. Nomenclature

$\mathbf{r}$	Field point coordinate.
$\mathbf{r}'$	Source point coordinate.
(ϵ, μ)	Material parameters of medium occupying V.
ω	Original problem frequency.
$(\mathbf{J}_\omega, \mathbf{M}_\omega)$	Electric and magnetic currents in original problem.
$(\rho_\omega, \rho_{m_\omega})$	Electric and magnetic charges in original problem.
$(\mathbf{E}_\omega, \mathbf{H}_\omega)$, $(\mathbf{D}_\omega, \mathbf{B}_\omega)$	Frequency-domain field in original problem.
ω_0	Shifted frequency.
$\mathbf{G}, \mathbf{F}$	Vector functions used in Green's theorem.
ψ	Wave function choice for expansion of $\mathbf{E}_\omega(\mathbf{r})$.
$\mathbf{a}$	Constant vector multiplying ψ.
k_0	Wavenumber in free space at ω_0.
k_s	Wavenumber in material in V at ω.
$\hat{\mathbf{n}}_i$	Surface normal on the boundaries directed into V.
$\hat{\mathbf{n}}$	Surface normal on the boundaries directed out of V.
$(\mathbf{J}^v_{\omega_0}, \mathbf{M}^v_{\omega_0})$	Equivalent electric and magnetic volume currents at ω_0.
$(\rho_{\omega 0}, \rho_{m\omega 0})$	Electric and magnetic charges in the equivalent problem.
$(\mathbf{J}^s_{\omega_0}, \mathbf{M}^s_{\omega_0})$	Equivalent electric and magnetic surface currents at ω_0.
$(\mathbf{E}_{\omega_0}, \mathbf{H}_{\omega_0})$	Frequency-domain field at shifted frequency.
$\mathbf{A}^v_{\omega_0}$	Magnetic vector potential due to $\mathbf{J}^v_{\omega_0}$.
$\mathbf{A}^s_{\omega_0}$	Magnetic vector potential due to $\mathbf{J}^s_{\omega_0}$.
$\mathbf{F}^v_{\omega_0}$	Electric vector potential due to $\mathbf{J}^v_{\omega_0}$.
$\mathbf{F}^s_{\omega_0}$	Electric vector potential due to $\mathbf{J}^s_{\omega_0}$.
$\mathbf{M}^{vA}_{\omega_0}$	$\mathbf{M}^v_{\omega_0} - \mathbf{M}_\omega$.

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II. Introduction

THIS PAPER presents a proof of an internal equivalence in the frequency domain at a shifted frequency for an electromagnetic problem involving a different frequency.

The original problem involves a lossless inhomogeneous region of space V, which has continuously differentiable material parameters. The sources in V, both electric and magnetic, radiate at frequency ω.

In the equivalent problem, the sources exist only in V and on its boundary S, and they radiate in free space at a different frequency ω_0. These equivalent sources consist of the sources of the original problem, and electric- and magnetic-type volume and surface currents. Equivalent volume currents are functions of the two frequencies ω and ω_0, and the sources and the field of the original problem, whereas the surface currents depend on the original field $(\mathbf{E}_\omega, \mathbf{H}_\omega)$ only.

The equivalence is presented in Section II, and its proof is given in Section III, followed by the conclusion in Section IV. The proof is based on an application of a vector Green's theorem on $(\mathbf{E}_\omega, \mathbf{H}_\omega)$, and on the potential formulation of the field in the equivalent problem, which is denoted by $(\mathbf{E}_{\omega_0}, \mathbf{H}_{\omega_0})$. The expansion is very similar to the Stratton–Chu formulation [1], [2], with the main difference being the choice of the wave function ψ.

In this paper, the word equivalence is meant for the equivalence of $(\mathbf{E}_{\omega_0}, \mathbf{H}_{\omega_0})$ to $(\mathbf{E}_\omega, \mathbf{H}_\omega)$ only. As will become obvious after the equivalence is stated and its proof is given, $\mathbf{D}$ and $\mathbf{B}$ fields in the original and equivalent problem are different.

III. Shifted-Frequency Internal Equivalence

Consider the region V illustrated in Fig. 1, bounded by the surfaces $S_1, \cdots, S_n$. The material parameters (ϵ, μ) in V are assumed to be continuously differentiable real functions of position. The electromagnetic field in V generated by the sources operating at the frequency ω satisfies the following

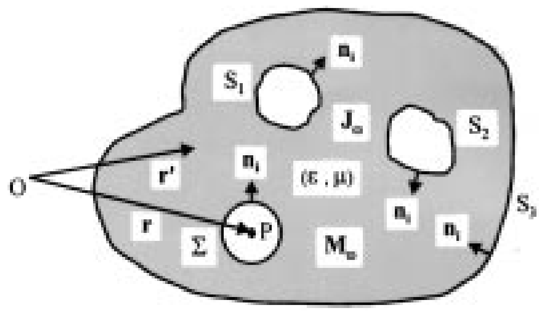

Fig. 1. Electromagnetic field in V is excited by the sources $(\mathbf{J}_\omega,\, \mathbf{M}_\omega)$ operating at the frequency ω. Field point P is surrounded by a sphere Σ for the application of vector Green's theorem.

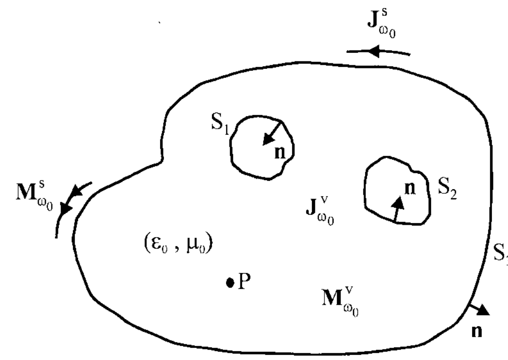

Fig. 2. Shifted-frequency equivalent problem for Fig. 1. Material body is replaced by the equivalent sources $(\mathbf{J}^v_{\omega_0},\, \mathbf{M}^v_{\omega_0})$ and $(\mathbf{J}^s_{\omega_0},\, \mathbf{M}^s_{\omega_0})$ radiating in free space at frequency ω.

Maxwell's equations:

$$\nabla \times \mathbf{E}_\omega = -\jmath\omega\mu\mathbf{H}_\omega - \mathbf{M}_\omega \tag{1}$$
$$\nabla \times \mathbf{H}_\omega = \jmath\omega\epsilon\mathbf{E}_\omega + \mathbf{J}_\omega \tag{2}$$
$$\nabla \cdot \mathbf{B}_\omega = \rho_{m\omega} \tag{3}$$
$$\nabla \cdot \mathbf{D}_\omega = \rho_\omega \tag{4}$$

and the following continuity equations:

$$\nabla \cdot \mathbf{J}_\omega + \jmath\omega\rho_\omega = 0 \tag{5}$$
$$\nabla \cdot \mathbf{M}_\omega + \jmath\omega\rho_{m\omega} = 0. \tag{6}$$

The electromagnetic field $(\mathbf{E}_\omega,\, \mathbf{H}_\omega)$ in V can be obtained in an equivalent problem where the sources radiate at a different frequency. The equivalence is given as follows.

Equivalence: The frequency-domain electromagnetic field $(\mathbf{E}_\omega,\, \mathbf{H}_\omega)$ in the region with properties described above and shown in Fig. 1 can be obtained using the following sources defined in V, radiating in free-space at the frequency ω_0:

$$\mathbf{J}^v_{\omega_0} = \jmath(\omega\epsilon - \omega_0\epsilon_0)\mathbf{E}_\omega + \mathbf{J}_\omega \tag{7}$$
$$\mathbf{M}^v_{\omega_0} = \jmath(\omega\mu - \omega_0\mu_0)\mathbf{H}_\omega + \mathbf{M}_\omega \tag{8}$$
$$\mathbf{J}^s_{\omega_0} = -\hat{\mathbf{n}} \times \mathbf{H}_\omega \tag{9}$$
$$\mathbf{M}^s_{\omega_0} = \hat{\mathbf{n}} \times \mathbf{E}_\omega \tag{10}$$

where $\hat{\mathbf{n}}$ is the surface normal on the boundary of V, and it is directed out of region V. The equivalent problem thus obtained is illustrated in Fig. 2. With the equivalent source definitions given in (7)–(10), the electromagnetic field in the equivalent problem satisfies the following Maxwell's equations:

$$\nabla \times \mathbf{E}_{\omega_0} = -\jmath\omega_0\mu_0\mathbf{H}_{\omega_0} - \mathbf{M}^v_{\omega_0} \tag{11}$$
$$\nabla \times \mathbf{H}_{\omega_0} = \jmath\omega_0\epsilon_0\mathbf{E}_{\omega_0} + \mathbf{J}^v_{\omega_0} \tag{12}$$
$$\nabla \cdot \mathbf{B}_{\omega_0} = \rho_{m\omega 0} \tag{13}$$
$$\nabla \cdot \mathbf{D}_{\omega_0} = \rho_{\omega 0} \tag{14}$$

and the following continuity equations:

$$\nabla \cdot \mathbf{J}^v_{\omega_0} + \jmath\omega_0\rho_{\omega 0} = 0 \tag{15}$$
$$\nabla \cdot \mathbf{M}^v_{\omega_0} + \jmath\omega_0\rho_{m\omega 0} = 0. \tag{16}$$

As a result, in order to prove shifted frequency internal equivalence (SFIE), we need to show that

$$\mathbf{E}_{\omega_0} = \mathbf{E}_\omega$$
$$\mathbf{H}_{\omega_0} = \mathbf{H}_\omega.$$

IV. Proof of SFIE

The proof of the equivalence will be carried out in four steps.

In the first step, an expansion of $\mathbf{E}_\omega(\mathbf{r})$ will be obtained using a vector Green's theorem. The procedure follows the same lines as the Stratton–Chu formulation [1], [2]. The main differences are the assumed inhomogeneity of the medium, the usage of source point-field point notation, and the usage of a different ψ function to carry out the expansion. A ψ function involving the wavenumber in free space at frequency ω_0 is chosen to obtain an expansion that can be compared to potential formulation results.

In the second step, the electric-field phasor of equivalent volume sources operating at ω_0 will be obtained in the frequency domain using potential theory.

The third step involves the calculation of the electric-field phasor of equivalent surface sources operating at ω_0 using the potential theory.

In the fourth and final step, the electric fields calculated in the second and the third step will be added to obtain $\mathbf{E}_{\omega_0}$, and the result will be shown to be the same as the expansion obtained for $\mathbf{E}_\omega$ in the first step. The discussion for the case of the magnetic-field phasor $\mathbf{H}_\omega(\mathbf{r})$ will complete the proof.

A. Step I: Expansion of $\mathbf{E}_\omega(\mathbf{r})$

The expansion of $\mathbf{E}_\omega(\mathbf{r})$ will be carried out using a vector Green's theorem. Consider again the region V, shown in Fig. 1, bounded by the surfaces $S_1, \cdots, S_n$. Use S to denote the union of all S_i, $i = 1, \cdots, n$. Let $\mathbf{F}$ and $\mathbf{G}$ be two vector functions of position in this region, with continuous first and second derivatives everywhere within V and on the boundary surfaces. Then, if $\hat{\mathbf{n}}_i$ is the unit vector normal to a bounding surface directed into the region V

$$\int_V (\mathbf{G} \cdot \nabla' \times \nabla' \times \mathbf{F} - \mathbf{F} \cdot \nabla' \times \nabla' \times \mathbf{G})\, dV = -\int_S (\mathbf{F} \times \nabla' \times \mathbf{G} - \mathbf{G} \times \nabla' \times \mathbf{F}) \cdot \hat{\mathbf{n}}_i\, dS. \tag{17}$$

In order to apply the given vector Green's theorem to expand $\mathbf{E}_\omega(\mathbf{r})$, consider an interior point P in V and surround this point by a sphere Σ of radius r_0 and denote the remaining volume by $V - V_\Sigma$. Considering Fig. 1, $V - V_\Sigma$ is the region bounded by S and S_Σ.

Choose the vector functions of position $\mathbf{G}$ and $\mathbf{F}$ as

$$\mathbf{G} = \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathbf{a} = \psi\mathbf{a} \tag{18}$$

$$\mathbf{F} = \mathbf{E}_\omega(\mathbf{r}') \tag{19}$$

where $k_0 = \omega_0\sqrt{\mu_0\epsilon_0}$, the wavenumber in free space at ω_0, and $\mathbf{a}$ is a constant vector.

In the region $V - V_\Sigma$, $\mathbf{G}$ and $\mathbf{F}$, as defined by (18) and (19), satisfy the continuity requirements of the vector Green's theorem given in (17). Applying this theorem to chosen $\mathbf{F}$ and $\mathbf{G}$, we have

$$\int_{V-V_\Sigma} (\psi\mathbf{a}\cdot\nabla' \times \nabla' \times \mathbf{E}_\omega - \mathbf{E}_\omega \cdot \nabla' \times \nabla' \times \psi\mathbf{a})\, dV = -\int_{S+S_\Sigma} (\mathbf{E}_\omega \times \nabla' \times \psi\mathbf{a} - \psi\mathbf{a} \times \nabla' \times \mathbf{E}_\omega)\cdot\hat{\mathbf{n}}_i\, dS. \tag{20}$$

In (20), and in the equations following, during the expansion it is understood that (ϵ, μ), ($\mathbf{E}_\omega$, $\mathbf{H}_\omega$), and ($\mathbf{J}_\omega$, $\mathbf{M}_\omega$) inside the integral signs depend on $\mathbf{r}'$, and this dependence is suppressed for brevity unless it is necessary to differentiate between the source and the field coordinates.

We can use (1) and (2) after converting the equations to primed coordinates to obtain $\nabla' \times \nabla' \times \mathbf{E}_\omega$ as

$$\nabla' \times \nabla' \times \mathbf{E}_\omega = k_s^2\mathbf{E}_\omega - j\omega\mu\mathbf{J}_\omega - \nabla' \times \mathbf{M}_\omega - j\omega\nabla'\mu \times \mathbf{H}_\omega \tag{21}$$

where $k_s = \omega\sqrt{\mu\epsilon}$ is the wavenumber at frequency ω for the material occupying the region shown in Fig. 1. Since ψ satisfies the scalar Helmholtz equation, one can easily show that

$$\nabla' \times \nabla' \times \psi\mathbf{a} = \nabla'(\mathbf{a}\cdot\nabla'\psi) + k_0^2\psi\mathbf{a}. \tag{22}$$

Using this together with (21), we obtain

$$\begin{aligned}\psi\mathbf{a}\cdot\nabla' \times \nabla' \times \mathbf{E}_\omega - \mathbf{E}_\omega\cdot\nabla' \times \nabla' \times \psi\mathbf{a} &= \mathbf{a}\cdot(-j\omega\mu\mathbf{J}_\omega\psi - \psi\nabla' \times \mathbf{M}_\omega + k_s^2\mathbf{E}_\omega\psi \\ &\quad - j\omega\psi\nabla'\mu \times \mathbf{H}_\omega) - \mathbf{E}_\omega\cdot[k_0^2\psi\mathbf{a} + \nabla'(\mathbf{a}\cdot\nabla'\psi)].\end{aligned} \tag{23}$$

Also noting that

$$\mathbf{E}_\omega\cdot\nabla'(\mathbf{a}\cdot\nabla'\psi) = \nabla'\cdot[\mathbf{E}_\omega(\mathbf{a}\cdot\nabla'\psi)] - (\mathbf{a}\cdot\nabla'\psi)\nabla'\cdot\mathbf{E}_\omega \tag{24}$$

$$\psi\nabla' \times \mathbf{M}_\omega = \nabla' \times (\mathbf{M}_\omega\psi) + \mathbf{M}_\omega \times \nabla'\psi \tag{25}$$

and combining all these facts, (20) can be rewritten as

$$\begin{aligned}&\mathbf{a}\cdot\int_{V-V_\Sigma} [(k_0^2 - k_s^2)\mathbf{E}_\omega\psi + j\omega\psi\nabla'\mu \times \mathbf{H}_\omega \\ &\qquad + j\omega\mu\mathbf{J}_\omega\psi + \mathbf{M}_\omega \times \nabla'\psi]\, dV \\ &\quad + \mathbf{a}\cdot\int_{V-V_\Sigma} \nabla' \times (\psi\mathbf{M}_\omega)\, dV + \int_{V-V_\Sigma} \nabla' \\ &\quad \cdot[\mathbf{E}_\omega(\mathbf{a}\cdot\nabla'\psi)]\, dV - \mathbf{a}\cdot\int_{V-V_\Sigma} (\nabla'\cdot\mathbf{E}_\omega)\nabla'\psi\, dV \\ &\qquad = \int_{S+S_\Sigma} [(\mathbf{E}_\omega \times \nabla' \times \psi\mathbf{a})\cdot\hat{\mathbf{n}}_i - (\psi\mathbf{a} \times \nabla' \times \mathbf{E}_\omega)\cdot\hat{\mathbf{n}}_i]\, dS.\end{aligned} \tag{26}$$

The second- and third-volume integrals in (26) can be transformed into the following surface integrals:

$$\mathbf{a}\cdot\int_{V-V_\Sigma} \nabla' \times (\psi\mathbf{M}_\omega)\, dV = -\mathbf{a}\cdot\int_{S+S_\Sigma} \psi\hat{\mathbf{n}}_i \times \mathbf{M}_\omega\, dS \tag{27}$$

$$\int_{V-V_\Sigma} \nabla'\cdot[\mathbf{E}_\omega(\mathbf{a}\cdot\nabla'\psi)]\, dV = -\mathbf{a}\cdot\int_{S+S_\Sigma} (\hat{\mathbf{n}}_i\cdot\mathbf{E}_\omega)\nabla'\psi\, dS. \tag{28}$$

In order to convert the surface integrals on the right-hand side in (26) to the desired form, the following transformations are utilized:

$$\begin{aligned}&[\mathbf{E}_\omega \times (\nabla' \times \psi\mathbf{a})]\cdot\hat{\mathbf{n}}_i \\ &\quad = [\mathbf{E}_\omega \times (\nabla'\psi \times \mathbf{a})]\cdot\hat{\mathbf{n}}_i = [(\hat{\mathbf{n}}_i \times \mathbf{E}_\omega) \times \nabla'\psi]\cdot\mathbf{a}\end{aligned} \tag{29}$$

$$\begin{aligned}&\psi(\mathbf{a} \times \nabla' \times \mathbf{E}_\omega)\cdot\hat{\mathbf{n}}_i \\ &\quad = -j\omega\mu\psi(\mathbf{a} \times \mathbf{H}_\omega)\cdot\hat{\mathbf{n}}_i - \psi(\mathbf{a} \times \mathbf{M}_\omega)\cdot\hat{\mathbf{n}}_i \\ &\quad = j\omega\mu\psi\mathbf{a}\cdot(\hat{\mathbf{n}}_i \times \mathbf{H}_\omega) + \psi\mathbf{a}\cdot(\hat{\mathbf{n}}_i \times \mathbf{M}_\omega).\end{aligned} \tag{30}$$

Now, collecting all these results, (26) can be given the desired form with all terms having $\mathbf{a}$ as an inner product. Since $\mathbf{a}$ is a constant vector, the result must be valid for all $\mathbf{a}$ and, therefore, it may be cancelled. After this cancellation and treating the surface integral terms for S_Σ and S separately, the equation is obtained as

$$\begin{aligned}&\int_{S_\Sigma} [-j\omega\mu\psi(\hat{\mathbf{n}}_i \times \mathbf{H}_\omega) + (\hat{\mathbf{n}}_i \times \mathbf{E}_\omega) \times \nabla'\psi + (\hat{\mathbf{n}}_i\cdot\mathbf{E}_\omega)\nabla'\psi]\, dS \\ &\quad = \int_{V-V_\Sigma} [j\omega\mu\mathbf{J}_\omega\psi + \mathbf{M}_\omega \times \nabla'\psi + j\omega\psi\nabla'\mu \times \mathbf{H}_\omega \\ &\qquad - (\nabla'\cdot\mathbf{E}_\omega)\nabla'\psi + (k_0^2 - k_s^2)\mathbf{E}_\omega\psi]\, dV \\ &\qquad - \int_S [-j\omega\mu\psi(\hat{\mathbf{n}}_i \times \mathbf{H}_\omega) + (\hat{\mathbf{n}}_i \times \mathbf{E}_\omega) \\ &\qquad \times \nabla'\psi + (\hat{\mathbf{n}}_i\cdot\mathbf{E}_\omega)\nabla'\psi]\, dS.\end{aligned} \tag{31}$$

The function ψ has the required singularity at the field point which is surrounded by the surface S_Σ. Considering the integral over S_Σ, we have the following for points on S_Σ:

$$\nabla'\psi = -\left(\jmath k_0 + \frac{1}{r_0}\right)\frac{e^{-\jmath k_0 r_0}}{r_0}\,\hat{\mathbf{n}}_i. \tag{32}$$

As the sphere Σ shrinks to zero, the first term in the integral over S_Σ in (31) tends to zero since the fields are finite at the field point P. The second and third integrals of the same equation result in -4π times the field value at P—exactly in the same manner as in the Stratton–Chu formulation [1], [2]. Therefore,

$$\lim_{r_0\to 0}\int_{S_\Sigma}[\cdot] = -4\pi\mathbf{E}_\omega(\mathbf{r}). \tag{33}$$

The final expression for $\mathbf{E}_\omega$, after substituting $\hat{\mathbf{n}}_i = -\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the normal to all bounding surfaces, but directed out of volume V is

$$\begin{aligned}\mathbf{E}_\omega(\mathbf{r}) = &-\frac{1}{4\pi}\int_V [(k_0^2 - k_s^2)\mathbf{E}_\omega\psi + \jmath\omega\psi\nabla'\mu \\ &\times \mathbf{H}_\omega - (\nabla'\cdot\mathbf{E}_\omega)\nabla'\psi]\,dV \\ &- \frac{1}{4\pi}\int_V [\jmath\omega\mu\mathbf{J}_\omega\psi + \mathbf{M}_\omega\times\nabla'\psi]\,dV \\ &- \frac{1}{4\pi}\int_S[-\jmath\omega\mu(\hat{\mathbf{n}}\times\mathbf{H}_\omega)\psi + (\hat{\mathbf{n}}\times\mathbf{E}_\omega) \\ &\times\nabla'\psi + (\hat{\mathbf{n}}\cdot\mathbf{E}_\omega)\nabla'\psi]\,dS\end{aligned} \tag{34}$$

where the volume integral over V is a principal value integral because of the limiting process for Σ.

B. Step II: Determination of the Fields of Equivalent Volume Sources $\mathbf{J}^v_{\omega_0}$ and $\mathbf{M}^v_{\omega_0}$

Let $\mathbf{E}^v_{\omega_0}(\mathbf{r})$ denote the phasor field produced by $\mathbf{J}^v_{\omega_0}$ and $\mathbf{M}^v_{\omega_0}$ at the field point $\mathbf{r}$ inside V. $\mathbf{J}^v_{\omega_0}$ and $\mathbf{M}^v_{\omega_0}$ are the equivalent volume currents operating at ω_0 given in (7) and (8), and are repeated here for easy reference:

$$\begin{aligned}\mathbf{J}^v_{\omega_0} &= \jmath(\omega\epsilon - \omega_0\epsilon_0)\mathbf{E}_\omega + \mathbf{J}_\omega \\ \mathbf{M}^v_{\omega_0} &= \jmath(\omega\mu - \omega_0\mu_0)\mathbf{H}_\omega + \mathbf{M}_\omega.\end{aligned}$$

$\mathbf{E}^v_{\omega_0}(\mathbf{r})$ can be found using magnetic- and electric-vector potentials $\mathbf{A}^v_{\omega_0}(\mathbf{r})$ and $\mathbf{F}^v_{\omega_0}(\mathbf{r})$, respectively, as

$$\begin{aligned}&\mathbf{E}^v_{\omega_0}(\mathbf{r}) \\ &\quad= -\jmath\omega_0\mathbf{A}^v_{\omega_0}(\mathbf{r}) - \frac{\jmath}{\omega_0\mu_0\epsilon_0}\nabla\nabla\cdot\mathbf{A}^v_{\omega_0}(\mathbf{r}) - \frac{1}{\epsilon_0}\nabla\times\mathbf{F}^v_{\omega_0}(\mathbf{r}).\end{aligned} \tag{35}$$

The individual terms in this equation will be investigated separately. Consider $\mathbf{A}^v_{\omega_0}(\mathbf{r})$, which is given by

$$\mathbf{A}^v_{\omega_0}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \mathbf{J}^v_{\omega_0}(\mathbf{r}')\psi\,dV \tag{36}$$

where the integral has to be evaluated in the principal-value sense. For brevity, the functional dependence of the sources inside the integrals on $\mathbf{r}'$, and of the results on $\mathbf{r}$, will be omitted in the following equations.

The ∇ operator for the divergence of $\mathbf{A}^v_{\omega_0}(\mathbf{r})$ can be brought safely inside the integral sign to obtain

$$\nabla\cdot\mathbf{A}^v_{\omega_0} = \frac{\mu_0}{4\pi}\int_V \mathbf{J}^v_{\omega_0}\cdot\nabla\psi\,dV = -\frac{\mu_0}{4\pi}\int_V \mathbf{J}^v_{\omega_0}\cdot\nabla'\psi\,dV. \tag{37}$$

Using

$$-\mathbf{J}^v_{\omega_0}\cdot\nabla'\psi = (\nabla'\cdot\mathbf{J}^v_{\omega_0})\psi - \nabla'\cdot(\mathbf{J}^v_{\omega_0}\psi) \tag{38}$$

in (37) and applying the divergence theorem, we obtain

$$\nabla\cdot\mathbf{A}^v_{\omega_0} = \frac{\mu_0}{4\pi}\int_V(\nabla'\cdot\mathbf{J}^v_{\omega_0})\psi\,dV - \frac{\mu_0}{4\pi}\int_S(\hat{\mathbf{n}}\cdot\mathbf{J}^v_{\omega_0})\psi\,dS. \tag{39}$$

Now the second ∇ operator for the gradient can be taken inside the integrals. Doing so, and using the fact that $\nabla\psi = -\nabla'\psi$, we get

$$\begin{aligned}&\nabla\nabla\cdot\mathbf{A}^v_{\omega_0} \\ &\quad= -\frac{\mu_0}{4\pi}\int_V(\nabla'\cdot\mathbf{J}^v_{\omega_0})\nabla'\psi\,dV + \frac{\mu_0}{4\pi}\int_S(\hat{\mathbf{n}}\cdot\mathbf{J}^v_{\omega_0})\nabla'\psi\,dS.\end{aligned} \tag{40}$$

A similar analysis will be carried out for $\mathbf{F}^v_{\omega_0}(\mathbf{r})$:

$$\begin{aligned}&-\frac{4\pi}{\epsilon_0}\nabla\times\mathbf{F}^v_{\omega_0} \\ &\quad= -\nabla\times\int_V \mathbf{M}^v_{\omega_0}\psi\,dV \\ &\quad= \int_V \nabla'\psi\times\mathbf{M}^v_{\omega_0}\,dV \qquad (41) \\ &\quad= \int_V \nabla'\times(\mathbf{M}^v_{\omega_0}\psi)\,dV - \int_V(\nabla'\times\mathbf{M}^v_{\omega_0})\psi\,dV \\ &\quad= \int_S(\hat{\mathbf{n}}\times\mathbf{M}^v_{\omega_0})\psi\,dS - \int_V(\nabla'\times\mathbf{M}^v_{\omega_0})\psi\,dV. \qquad (42)\end{aligned}$$

At this point, divide $\mathbf{M}^v_{\omega_0}$ into two parts as

$$\mathbf{M}^v_{\omega_0} = \mathbf{M}^{vA}_{\omega_0} + \mathbf{M}_\omega \tag{43}$$

where $\mathbf{M}^{vA}_{\omega_0} = \jmath(\omega\mu - \omega_0\mu_0)\mathbf{H}_\omega$, and use (41) for finding the contribution of $\mathbf{M}_\omega$ and (42) for that of $\mathbf{M}^{vA}_{\omega_0}$. Carrying out these calculations, the $\mathbf{E}^v_{\omega_0}(\mathbf{r})$ expression given in (35) becomes

$$\begin{aligned}&\mathbf{E}^v_{\omega_0}(\mathbf{r}) \\ &\quad= \frac{-\jmath\omega_0\mu_0}{4\pi}\int_V \mathbf{J}^v_{\omega_0}\psi\,dV - \frac{1}{4\pi}\int_V \mathbf{M}_\omega \\ &\qquad\times\nabla'\psi\,dV + \frac{\jmath}{4\pi\omega_0\epsilon_0}\int_V(\nabla'\cdot\mathbf{J}^v_{\omega_0})\nabla'\psi\,dV \\ &\qquad- \frac{1}{4\pi}\int_V(\nabla'\times\mathbf{M}^{vA}_{\omega_0})\psi\,dV + \frac{1}{4\pi}\int_S(\hat{\mathbf{n}}\times\mathbf{M}^{vA}_{\omega_0})\psi\,dS \\ &\qquad- \frac{j}{4\pi\omega_0\epsilon_0}\int_S(\hat{\mathbf{n}}\cdot\mathbf{J}^v_{\omega_0})\nabla'\psi\,dS.\end{aligned} \tag{44}$$

The individual terms in (44) can be expanded as

$$\begin{aligned}\nabla' \times \mathbf{M}^{vA}_{\omega_0} &= \nabla' \times [j(\omega\mu - \omega_0\mu_0)\mathbf{H}_\omega] \\ &= j\omega\nabla'\mu \times \mathbf{H}_\omega + \omega\omega_0\mu_0\epsilon\mathbf{E}_\omega - k_s^2\mathbf{E}_\omega \\ &\quad + j\omega\mu\mathbf{J}_\omega - j\omega_0\mu_0\mathbf{J}_\omega \\ -j\omega_0\mu_0\mathbf{J}^v_{\omega_0} &= \omega\omega_0\epsilon\mu_0\mathbf{E}_\omega - k_0^2\mathbf{E}_\omega - j\omega_0\mu_0\mathbf{J}_\omega \\ \nabla' \cdot \mathbf{J}^v_{\omega_0} &= \nabla' \cdot [j\omega\epsilon\mathbf{E}_\omega - j\omega_0\epsilon_0\mathbf{E}_\omega + \mathbf{J}_\omega] \\ &= -j\omega_0\epsilon_0\nabla' \cdot \mathbf{E}_\omega \\ \hat{\mathbf{n}} \times \mathbf{M}^{vA}_{\omega_0} &= j(\omega\mu - \omega_0\mu_0)(\hat{\mathbf{n}} \times \mathbf{H}_\omega) \\ \hat{\mathbf{n}} \cdot \mathbf{J}^v_{\omega_0} &= j(\omega\epsilon - \omega_0\epsilon_0)(\hat{\mathbf{n}} \cdot \mathbf{E}_\omega) + (\hat{\mathbf{n}} \cdot \mathbf{J}_\omega).\end{aligned}$$

Using all these expansions of various terms of (44), we finally obtain

$$\begin{aligned}\mathbf{E}^v_{\omega_0}(\mathbf{r}) =& \frac{1}{4\pi}\int_V [(k_s^2 - k_0^2)\mathbf{E}_\omega\psi - j\omega\psi\nabla'\mu \\ &\times \mathbf{H}_\omega + (\nabla' \cdot \mathbf{E}_\omega)\nabla'\psi]\, dV \\ &+ \frac{1}{4\pi}\int_V [-j\omega\mu\mathbf{J}_\omega\psi - \mathbf{M}_\omega \times \nabla'\psi]\, dV \\ &- \frac{j}{4\pi\omega_0\epsilon_0}\int_S (\hat{\mathbf{n}} \cdot \mathbf{J}_\omega)\nabla'\psi\, dS \\ &+ \frac{1}{4\pi}\int_S j\omega\mu(\hat{\mathbf{n}} \times \mathbf{H}_\omega)\psi\, dS \\ &- \frac{1}{4\pi}\int_S j\omega_0\mu_0(\hat{\mathbf{n}} \times \mathbf{H}_\omega)\psi\, dS \\ &+ \frac{\omega}{4\pi\omega_0\epsilon_0}\int_S \epsilon(\hat{\mathbf{n}} \cdot \mathbf{E}_\omega)\nabla'\psi\, dS \\ &- \frac{1}{4\pi}\int_S (\hat{\mathbf{n}} \cdot \mathbf{E}_\omega)\nabla'\psi\, dS.\end{aligned} \tag{45}$$

C. Step III: Determination of the Fields of Equivalent Surface Sources $\mathbf{J}^s_{\omega_0}$ and $\mathbf{M}^s_{\omega_0}$

Let $\mathbf{E}^s_{\omega_0}(\mathbf{r})$ be the field radiated by the surface sources operating at ω_0. The equivalent surface sources are repeated here for convenience:

$$\begin{aligned}\mathbf{J}^s_{\omega_0} &= -\hat{\mathbf{n}} \times \mathbf{H}_\omega \\ \mathbf{M}^s_{\omega_0} &= \hat{\mathbf{n}} \times \mathbf{E}_\omega.\end{aligned}$$

$\mathbf{E}^s_{\omega_0}(\mathbf{r})$ can be written using potential formulation as

$$\begin{aligned}&\mathbf{E}^s_{\omega_0}(\mathbf{r}) \\ &\quad = -j\omega_0\mathbf{A}^s_{\omega_0}(\mathbf{r}) - \frac{j}{\omega_0\epsilon_0\mu_0}\nabla\nabla \cdot \mathbf{A}^s_{\omega_0}(\mathbf{r}) - \frac{1}{\epsilon_0}\nabla \times \mathbf{F}^s_{\omega_0}(\mathbf{r})\end{aligned} \tag{46}$$

where $\mathbf{A}^s_{\omega_0}(\mathbf{r})$ and $\mathbf{F}^s_{\omega_0}(\mathbf{r})$ are the magnetic- and electric-vector potentials, respectively. The electric-vector potential term can be written as

$$-\frac{1}{\epsilon_0}\nabla \times \mathbf{F}^s_{\omega_0}(\mathbf{r}) = -\frac{1}{4\pi}\int_S (\hat{\mathbf{n}} \times \mathbf{E}_\omega) \times \nabla'\psi \tag{47}$$

and the magnetic-vector potential term can be modified as

$$\begin{aligned}\mathbf{A}^s_{\omega_0}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_S (-\hat{\mathbf{n}} \times \mathbf{H}_\omega)\psi\, dS \\ &= \frac{\mu_0}{4\pi}\int_V -\nabla' \times (\mathbf{H}_\omega\psi)\, dV \\ &= \frac{\mu_0}{4\pi}\nabla \times \int_V \mathbf{H}_\omega\psi\, dV - \frac{\mu_0}{4\pi}\int_V j\omega\epsilon\mathbf{E}_\omega\psi\, dV \\ &\quad - \frac{\mu_0}{4\pi}\int_V \mathbf{J}_\omega\psi\, dV.\end{aligned} \tag{48}$$

When taking the divergence of $\mathbf{A}^s_{\omega_0}(\mathbf{r})$ given in (48), the first term gives zero result and we obtain

$$\nabla \cdot \mathbf{A}^s_{\omega_0}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V j\omega\epsilon\mathbf{E}_\omega \cdot \nabla'\psi\, dV + \frac{\mu_0}{4\pi}\int_V \mathbf{J}_\omega \cdot \nabla'\psi\, dV. \tag{49}$$

Using

$$\begin{aligned}\epsilon\mathbf{E}_\omega \cdot \nabla'\psi &= \nabla' \cdot [\epsilon\mathbf{E}_\omega\psi] - \psi\nabla' \cdot [\epsilon\mathbf{E}_\omega] \\ \mathbf{J}_\omega \cdot \nabla'\psi &= \nabla' \cdot (\mathbf{J}_\omega\psi) - \psi\nabla' \cdot \mathbf{J}_\omega\end{aligned} \tag{50}$$

in (49), noting that $\nabla \cdot \mathbf{D}_\omega = \rho_\omega$ and $\nabla \cdot \mathbf{J}_\omega = -j\omega\rho_\omega$ in V, and applying the divergence theorem gives

$$\nabla \cdot \mathbf{A}^s_{\omega_0} = \frac{\mu_0}{4\pi}\int_S j\omega\epsilon(\hat{\mathbf{n}} \cdot \mathbf{E}_\omega)\psi\, dS + \frac{\mu_0}{4\pi}\int_S (\hat{\mathbf{n}} \cdot \mathbf{J}_\omega)\psi\, dS \tag{51}$$

which allows taking the gradient operator inside the integral sign as

$$\begin{aligned}&\nabla\nabla \cdot \mathbf{A}^s_{\omega_0} \\ &\quad = -\frac{\mu_0}{4\pi}\int_S j\omega\epsilon(\hat{\mathbf{n}} \cdot \mathbf{E}_\omega)\nabla'\psi\, dS - \frac{\mu_0}{4\pi}\int_S \mathbf{J}_\omega \cdot \hat{\mathbf{n}}\nabla'\psi\, dS.\end{aligned} \tag{52}$$

Gathering these results in the $\mathbf{E}^s_{\omega_0}$ expression, we obtain

$$\begin{aligned}\mathbf{E}^s_{\omega_0}(\mathbf{r}) &= \frac{-j\omega_0\mu_0}{4\pi}\int_S (-\hat{\mathbf{n}} \times \mathbf{H}_\omega)\psi\, dS \\ &\quad - \frac{\omega}{4\pi\omega_0\epsilon_0}\int_S \epsilon(\hat{\mathbf{n}} \cdot \mathbf{E}_\omega)\nabla'\psi\, dS \\ &\quad + \frac{j}{4\pi\omega_0\epsilon_0}\int_S (\mathbf{J}_\omega \cdot \hat{\mathbf{n}})\nabla'\psi\, dS \\ &\quad - \frac{1}{4\pi}\int_S (\hat{\mathbf{n}} \times \mathbf{E}_\omega) \times \nabla'\psi\, dS.\end{aligned} \tag{53}$$

D. Step IV: Comparison of the Expansion of $\mathbf{E}_\omega(\mathbf{r})$ and Total Field $\mathbf{E}_{\omega_0}(\mathbf{r})$ Generated by the Shifted Frequency Sources $\mathbf{J}^v_{\omega_0}$, $\mathbf{M}^v_{\omega_0}$, $\mathbf{J}^s_{\omega_0}$, and $\mathbf{M}^s_{\omega_0}$

The total field phasor $\mathbf{E}_{\omega_0}(\mathbf{r})$ generated by the shifted frequency sources can be found by the addition of field-of-volume-type sources given in (45) and that of surface-type sources given in (53). Noting the cancellation of the first three surface integrals in (53) in the addition, the result is obtained as

$$\begin{aligned}\mathbf{E}_{\omega_0}(\mathbf{r}) = &-\frac{1}{4\pi}\int_V [(k_0^2 - k_s^2)\mathbf{E}_\omega\psi + \jmath\omega\psi\nabla'\mu \\ &\times \mathbf{H}_\omega - (\nabla'\cdot\mathbf{E}_\omega)\nabla'\psi]\,dV \\ &- \frac{1}{4\pi}\int_V [\jmath\omega\mu\mathbf{J}_\omega\psi + \mathbf{M}_\omega\times\nabla'\psi]\,dV \\ &- \frac{1}{4\pi}\int_S [-\jmath\omega\mu(\hat{\mathbf{n}}\times\mathbf{H}_\omega)\psi + (\hat{\mathbf{n}}\times\mathbf{E}_\omega) \\ &\times\nabla'\psi + (\hat{\mathbf{n}}\cdot\mathbf{E}_\omega)\nabla'\psi]\,dS. \qquad (54)\end{aligned}$$

However, this is identical with the expansion of $\mathbf{E}_\omega(\mathbf{r})$ given in (34) and, hence,

$$\mathbf{E}_{\omega_0}(\mathbf{r}) = \mathbf{E}_\omega(\mathbf{r}) \qquad (55)$$

is obtained proving the equivalence for the electric-field phasor.

A completely dual derivation following a similar approach may be followed to show that $\mathbf{H}_{\omega_0}(\mathbf{r}) = \mathbf{H}_\omega(\mathbf{r})$ in V also, which will not be repeated here.

For points on the surface S, 4π in all of the previous equations must be replaced by the correct subtended solid angle. For example, for smooth points on the surface S, the subtended solid angle is 2π [3], and all previous equations will be valid for such points once 4π is replaced with 2π in them. As a result, the equivalence is also valid for points on S.

V. Conclusion

The SFIE is introduced. It is shown that the frequency-domain field ($\mathbf{E}_\omega$, $\mathbf{H}_\omega$) in a lossless inhomogeneous region with sources radiating at frequency ω can be obtained using a set of equivalent volume and surface currents radiating in free space at a different frequency ω_0. Equivalent sources include the original sources, and they are functions of the two frequencies and the original field ($\mathbf{E}_\omega$, $\mathbf{H}_\omega$). It is worthwhile to mention again that the SFIE is valid for $\mathbf{E}$ and $\mathbf{H}$ fields only, but not for $\mathbf{D}$ and $\mathbf{B}$.

A direct application of this equivalence is that it can be used to construct an internal equivalence at a shifted frequency for the related electromagnetic scattering problem [3] if data are needed in a band of frequency. The equivalent volume currents defined in this paper are very similar in form to equivalent polarization and magnetization currents [4] and, hence, can be used in a similar manner in the computational methods for scattering [4], [5]. The advantage of using shifted-frequency equivalent sources is that ω_0 can be kept constant while the incident field frequency ω changes. As a result, the full computation of fields at each different frequency for volume-type equivalent sources can be avoided. The initial studies on the usage of the SFIE for multifrequency electromagnetic scattering are reported in [6] and [7], and will be discussed in a future paper.

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